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A result on Macaulay's curve

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Abstract

We are able to improve what is known about two assumed homogeneous polynomials cutting out Macaulay's curve $C_4 \subseteq \mathbb{P}^3_k$ set-theoretically, in characteristic zero. We use local cohomology and an idea from Thoma.

0 Introduction

Let k be an algebraically closed field of characteristic zero; let w, x, y, z be indeterminates and

$$\mathfrak{p} = \mathfrak{p}_{C_4} \subseteq k[w, x, y, z] =: R$$

the ideal of Macaulay's curve, that is the curve with parametrization $[s^4 : s^3t : st^3 : t^4]$ in $\mathbb{P}^3_k$. We assume throughout the paper that f and g are homogeneous polynomials of degrees d_1 resp. d_2 (in the usual sense) in $\mathfrak{p}$ that cut out p set-theoretically in the sense that $\sqrt{(f,g)R} = \mathfrak{p}$. It is a well-known and hard problem to find out whether such polynomials f and g exist or not (such polynomials do exist in positive characteristic: See [H] or [RS, II]). See [L] for a survey on set-theoretic complete intersections.

It is natural to endow R with the bigrading $\deg w = (4, 0), \deg x = (3, 1), \deg y = (1, 3), \deg z = (0, 4)$. The ideal p in R is bihomogeneous, where "bihomogeneous" here and in the following means "homogeneous with respect to the above bigrading". We decompose f and g as sums of their bihomogeneous components:

$$f = \sum_{i=i_{\min}}^{i_{\max}} f^{(i,4d_1-i)}, g = \sum_{j=j_{\min}}^{j_{\max}} g^{(j,4d_2-j)}$$

($f^{(i,4d_1-i)}$ is the homogeneous component of f of bidegree $(i, 4d_1 - i)$; similarly for g. $i_{\min}, i_{\max}, j_{\min}, j_{\max}$ are chosen such that $f_{\min} := f^{(i_{\min},4d_1-i_{\min})}, f_{\max} := f^{(i_{\max},4d_1-i_{\max})}, g_{\min} := g^{(j_{\min},4d_2-j_{\min})}, g_{\max} := g^{(j_{\max},4d_2-j_{\max})} \neq 0$). This situation was studied by Thoma (see e. g. [T4], [T3]):

(i) Both pairs $f_{\min}, g_{\min}$ and $f_{\max}, g_{\max}$ have a proper greatest common divisor in R, i. e. each of the ideals $(f_{\min}, g_{\min})R$ and $(f_{\max}, g_{\max})R$ is contained in a prime ideal of height one ([T4, Th. 2.10]).

(ii) One of those two greatest common divisors is contained in $\mathfrak{p}$ ([T4, Cor. 2.9]).

We are able to improve these results:

(i') For both ideals $(f_{\min}, g_{\min})R$ and $(f_{\max}, g_{\max})R$ it is true that *all* minimal prime divisors have height one, i. e. both ideals are principal up to radical (theorem 1.2.(a)).

(ii') The greatest common divisors of *both* pairs $f_{\min}, g_{\min}$ and $f_{\max}, g_{\max}$ are contained in $\mathfrak{p}$ (theorem 1.2.(b)).

For our proofs of (i') and (ii') we use local cohomology and a modification of the following idea from Thoma ([T4]):

Let λ and μ be additional indeterminates.

$$F(w,x,y,z,\lambda,\mu) := f(\lambda^4 w, \lambda^3\mu x, \lambda\mu^3 y, \mu^4 z) = \textstyle\sum_i f^{(i,4d_1-i)}\lambda^i\mu^{4d_1-i}$$
$$G(w,x,y,z,\lambda,\mu) := g(\lambda^4 w, \lambda^3\mu x, \lambda\mu^3 y, \mu^4 z) = \textstyle\sum_i g^{(i,4d_2-i)}\lambda^i\mu^{4d_2-i}$$

$\in R[\lambda,\mu] =: S$. There can be no point $[w_0 : x_0 : y_0 : z_0] \in \mathbb{P}^3_k \setminus C_4$ with $\lambda_0, \mu_0 \in k^*$ and such that

$$\underbrace{F(w_0,x_0,y_0,z_0,\lambda_0,\mu_0)}_{=f(\lambda_0^4 w_0, \lambda_0^3\mu_0 x_0, \lambda_0\mu_0^3 y_0, \mu_0^4 z_0)} = \underbrace{G(w_0,x_0,y_0,z_0,\lambda_0,\mu_0)}_{=g(\lambda_0^4 w_0, \lambda_0^3\mu_0 x_0, \lambda_0\mu_0^3 y_0, \mu_0^4 z_0)} = 0,$$

i. e.

$$[\lambda_0^4 w_0 : \lambda_0^3\mu_0 x_0 : \lambda_0\mu_0^3 y_0 : \mu_0^4 z_0]$$

belongs to $V(f,g)$, because such a point could not have the form $[s^4 : s^3t : st^3 : t^4]$ (otherwise – because of $\lambda_0, \mu_0 \neq 0$ – also $[w_0 : x_0 : y_0 : z_0]$ would have this form and would therefore belong to C_4).

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1 Results

We modify Thoma's observation which was described in the introduction: Recall that we assume that f and g are homogeneous polynimals of degrees d_1 resp. d_2 in $\mathfrak{p} = \mathfrak{p}_{C_4}$ that cut out p set-theoretically in the sense that $\sqrt{(f,g)R} = \mathfrak{p}$. Let λ be an additional indeterminate.

$$F_1(w,x,y,z,\lambda) := f(\lambda^4 w, \lambda^3 x, \lambda y, z) = \textstyle\sum_i f^{(i,4d_1-i)}\lambda^i$$
$$G_1(w,x,y,z,\lambda) := g(\lambda^4 w, \lambda^3 x, \lambda y, z) = \textstyle\sum_i g^{(i,4d_2-i)}\lambda^i$$

$\in R[\lambda] =: T$. There can be no point $[w_0 : x_0 : y_0 : z_0] \in \mathbb{P}^3_k \setminus C_4$, represented by $(w_0,x_0,y_0,z_0) \neq (0,0,0,0)$, and no $\lambda_0 \in k^*$ such that

$$\underbrace{F_1(w_0,x_0,y_0,z_0,\lambda_0)}_{=f(\lambda_0^4 w_0, \lambda_0^3 x_0, \lambda_0 y_0, z_0)} = \underbrace{G_1(w_0,x_0,y_0,z_0,\lambda_0)}_{=g(\lambda_0^4 w_0, \lambda_0^3 x_0, \lambda_0 y_0, z_0)} = 0,$$

i. e.

$$[\lambda_0^4 w_0 : \lambda_0^3 x_0 : \lambda_0 y_0 : z_0]$$

belongs to $V(f,g)$, because such a point could not have the form $[s^4 : s^3t : st^3 : t^4]$ (otherwise – because of $\lambda_0 \neq 0$ – also $[w_0 : x_0 : y_0 : z_0]$ would have this form and would therefore belong to C_4).

Remark 1.1. *All arguments in the sequel can be translated in an obvious way to*

$$F_2(w,x,y,z,\mu) := f(w,\mu x,\mu^3 y,\mu^4 z) = \sum_i f^{(i,4d_1-i)}\mu^{4d_1-i}$$
$$G_2(w,x,y,z,\mu) := g(w,\mu x,\mu^3 y,\mu^4 z) = \sum_i g^{(i,4d_2-i)}\mu^{4d_2-i}$$

$\in R[\mu]$ *(this time, of course, μ is the additional indeterminate); the obtained results are analogous with 'max' instead of 'min'.*

□

Note that the observation from the beginning of this section works equally for $\tilde{F}_1$ and $\tilde{G}_1$, where these two polynomials are obtained from F_1 and G_1 by cancelling out λ as much as possible. We claim that

$$\sqrt{(\tilde{F}_1,\tilde{G}_1)T} = \sqrt{(\lambda, f_{\min}, g_{\min})T} \cap \mathfrak{p}_{C_4}T \qquad (1)$$

(with $T = R[\lambda]$ as above).

"$\subseteq$" is trivial; "$\supseteq$": Let $P = (w_0, x_0, y_0, z_0, \lambda_0)$ be an arbitrary point on $V(\tilde{F}_1,\tilde{G}_1)$, we have to show that $P \in V(\lambda, f_{\min}, g_{\min}) \cup V(\mathfrak{p}_{C_4}T)$: In case $\lambda_0 = 0$, evaluation at P makes all bihomogeneous components of $\tilde{F}_1$ and of $\tilde{G}_1$ apart from "the minimal ones" vanish, therefore we have $P \in V(\lambda, f_{\min}, g_{\min})$; and the other case $\lambda_0 \neq 0$ follows from the observation from the beginning of this section.

We study the minimal prime divisors of $(\tilde{F}_1,\tilde{G}_1)T$, our main source of information is formula (1).

Let

$$\mathfrak{p}_1, \ldots, \mathfrak{p}_r, \mathfrak{q}_1, \ldots, \mathfrak{q}_s$$

be exactly the minimal prime divisors of $(f_{\min}, g_{\min})R$, where all $\mathfrak{p}_i$ have height one (these $\mathfrak{p}_i$ are therefore principal, they encode information on the $\gcd(f_{\min}, g_{\min})$) and all $\mathfrak{q}_i$ have height two. Clearly,

$$(\lambda,\mathfrak{p}_1), \ldots, (\lambda,\mathfrak{p}_r), (\lambda,\mathfrak{q}_1), \ldots, (\lambda,\mathfrak{q}_s)$$

are exactly the minimal prime divisors of $(\lambda, f_{\min}, g_{\min})T$.

The prime ideals $(\lambda,\mathfrak{p}_i)$ have height two and the prime ideals $(\lambda,\mathfrak{q}_i)$ have height three. We get

$$\sqrt{(\tilde{F}_1,\tilde{G}_1)T} = [(\lambda,\mathfrak{p}_1) \cap \ldots \cap (\lambda,\mathfrak{p}_r) \cap (\lambda,\mathfrak{q}_1) \cap \ldots \cap (\lambda,\mathfrak{q}_s)] \cap \mathfrak{p}_{C_4}T \qquad (2)$$

All prime ideals occurring in the bracket $[\ldots]$ contain λ, $\mathfrak{p}_{C_4}T$ does not contain λ.

In particular, between these $r+s+1$ prime ideals only one type of inclusion is possible: $\mathfrak{p}_{C_4}T$ can possibly be contained in some $(\lambda,\mathfrak{q}_i)$, equivalently: $\mathfrak{p}_{C_4}$ is contained in (and therefore equal to) some $\mathfrak{q}_i$.

- First case: $\mathfrak{p}_{C_4}$ is contained in no $\mathfrak{q}_i$: This means that between our $r+s+1$ prime ideals there are no inclusions at all. Since there are no inclusions, formula (2) is the (unique) minimal decomposition of $\sqrt{(\tilde{F}_1, \tilde{G}_1)T}$, the $r+s+1$ prime ideals are exactly the minimal prime divisors of $(\tilde{F}_1, \tilde{G}_1)T$. But the latter ideal has no minimal prime divisors of height three, i. e. $s = 0$. All minimal prime divisors of $(f_{\min}, g_{\min})R$ have height one.

- Second case: $\mathfrak{p}_{C_4}$ is contained in (and therefore equal to) some $\mathfrak{q}_i$: There is exactly one inclusion among the prime ideals in (2), namely $\mathfrak{p}_{C_4} \subseteq (\lambda, \mathfrak{q}_i)$; since this is the only inclusion, omitting $(\lambda, \mathfrak{q}_i)$ from (2) leads to the minimal decomposition of $\sqrt{(\tilde{F}_1, \tilde{G}_1)T}$. However, again, no minimal prime divisor of $(\tilde{F}_1, \tilde{G}_1)T$ has height three. Therefore, we must have $s = 1$, i. e. $(f_{\min}, g_{\min})R$ has exactly one minimal prime divisor of height two, namely $\mathfrak{p}_{C_4}$ (we will see below that this second case is actually impossible).

We summarize: The only minimal prime divisor of height two which $(f_{\min}, g_{\min})R$ can possibly have, is $\mathfrak{p}_{C_4}$. Furthermore, at least one minimal prime divisor of height one must exist, since otherwise the radical of $(f_{\min}, g_{\min})R$ would equal its (only) minimal prime divisor $\mathfrak{p}_{C_4}$, contradicting [T3, p. 816]. In particular, we may write (1) in the form

$$\sqrt{(\tilde{F}_1, \tilde{G}_1)T} = \sqrt{(\lambda, t_{\min})T} \cap \mathfrak{p}_{C_4}T. \tag{3}$$

with $t_{\min}$ the greatest common divisor of $f_{\min}$ and $g_{\min}$ ($t_{\min}$ is not a unit since $(f_{\min}, g_{\min})R$ has a minimal prime divisor of height one).

Theorem 1.2. *(a) All minimal prime divisors of $(f_{min}, g_{min})R$ and of $(f_{max}, g_{max})R$ have height one. In particular, both pairs f_{min}, g_{min} and f_{max}, g_{max} have a proper (non-unit) common divisor t_{min} resp. t_{max} in R.*

(b) Both t_{min} and t_{max} are contained in $\mathfrak{p}_{C_4}$.

(c)

$$\sqrt{(\tilde{F}, \tilde{G})S} = (\lambda, \mu) \cap \sqrt{(\lambda, t_{min})S} \cap \sqrt{(\mu, t_{max})S} \cap \mathfrak{p}_{C_4}S. \tag{4}$$

Proof: (a) and (b): The ring $S_1 = k[\lambda]_{(\lambda)}[w, x, y, z] = R_0[w, x, y, z]$ with R_0 the subring of elements of degree zero and $\deg w, x, y, z = 1$ is graded and *local (in particular: noetherian), using terminology from [BS]. It is also a localization of T. The ring $\overline{S_1} := S_1/(\tilde{F}_1, \tilde{G}_1)S_1$ is also *local and we can formulate (3) for its ideals $a := \sqrt{(\lambda, t_{\min})\overline{S_1}}$ and $b := \mathfrak{p}_{C_4}\overline{S_1}$. (3) says that ab is nilpotent, however a and b are non-nilpotent (this is clear from the discussion preceeding (3)). The following trick is known, to the best of our knowledge it goes back to Irving Kaplansky (see also [H, Prop. 2.1]): The (exact) Mayer-Vietoris sequence for local cohomology of $\overline{S_1}$ with respect to a and b starts as follows:

$$0 \to \Gamma_{a+b}(\overline{S_1}) \to \Gamma_a(\overline{S_1}) \oplus \Gamma_b(\overline{S_1}) \to \Gamma_{a\cap b}\overline{S_1} = \overline{S_1} \to H^1_{a+b}(\overline{S_1}). \tag{$*$}$$

Therefore, depth$(a+b,\overline{S_1})$ must be at most one, because otherwise $(*)$ would provide an isomorphism

$$\Gamma_a(\overline{S_1}) \oplus \Gamma_b(\overline{S_1}) \cong \overline{S_1},$$

which is impossible since $\overline{S_1}$ is *local. In the ring S_1 this means that the depth and hence also the height of $\sqrt{(\lambda, t_{\min})S_1} + \mathfrak{p}_{C_4}S_1$ is at most $1+2=3$ (note that $\tilde{F}_1, \tilde{G}_1$ is a regular sequence in S_1, e. g. by 3). But this is only possibly if $t_{\min}$ is in p_{C_4}.

Now, both $(f_{\min}, g_{\min})R$ and $(f_{\max}, g_{\max})R$ have a minimal prime divisor of height one and which is contained in $\mathfrak{p}_{C_4}$; therefore $\mathfrak{p}_{C_4}$, which is – as we have seen above – the only possible minimal prime divisor of height two, cannot be minimal. Consequently, all minimal prime divisors of $(f_{\min}, g_{\min})R$ (analougously, of $(f_{\max}, g_{\max})R$) have height one.

(c) We work with the polynomials F and G from the introduction. The observation from the end of the introduction works equally for $\tilde{F}$ and $\tilde{G}$, where these two polynomials are obtained from F and G by cancelling out λ and μ as much as possible. We claim that

$$\sqrt{(\tilde{F}, \tilde{G})S} = (\lambda, \mu) \cap \sqrt{(\lambda, f_{\min}, g_{\min})S} \cap \sqrt{(\mu, f_{\max}, g_{\max})S} \cap \mathfrak{p}_{C_4}S. \qquad (5)$$

"$\subseteq$" follows from the fact that both f and g consist of at least two bihomogeneous components ([T3, p. 816], [T3, Lemma 3.1]); "$\supseteq$": Let $P = (w_0, x_0, y_0, z_0, \lambda_0, \mu_0)$ be an arbitrary point on $V(\tilde{F}, \tilde{G})$, we have to show that $P \in V(\lambda, \mu) \cup V(\lambda, f_{\min}, g_{\min}) \cup V(\mu, f_{\max}, g_{\max}) \cup V(\mathfrak{p}_{C_4}S)$: The case $\lambda_0, \mu_0 = 0$ is trivial; case $\lambda_0 = 0, \mu_0 \neq 0$: Evaluation at P makes all bihomogeneous components of $\tilde{F}$ and of $\tilde{G}$ apart from "the minimal ones" vanish, therefore we have $P \in V(\lambda, f_{\min}, g_{\min})$; the case $\mu_0 = 0, \lambda_0 \neq 0$ is analogous with "maximal" instead of "minimal"; finally the case $\lambda_0 \neq 0, \mu_0 \neq 0$ follows from Thoma's idea described in the introduction.

□

Remark 1.3. *It is clear that $\lambda \cdot t_{max}$ (and, similarly, $\mu \cdot t_{min}$) belongs to all four ideals in the right hand side of formula (3) and, therefore, a power of it can be written as a linear combination of $\tilde{F}$ and $\tilde{G}$. Similarly, t_{min} is in the radical of (f_{min}, g_{min}) and t_{max} is in the radical of (f_{max}, g_{max}).*

Remark 1.4. *The non-trivial result from Thoma that the number of (non-zero) bihomogenous components of f or of g is at least three ([T4, Th. 3.1.(a)]) follows immdediately from theorem 1.2 b) together with the well-known fact that the number of (non-zero) bihomogenous components of f or of g is at least two ([T3, p. 816]). (Is also well-known and easy to see that neither f nor g can be bihomogenous ([T3, Lemma 3.1])).*

Remark 1.5. *Finally note that all our results and proofs immediately generalize to arbitrary symmetric, non-arithmetically Cohen Macualay monomial curves $[s^d : s^a t^b : s^b t^a : t^d]$ (see [T3, p. 816]).*

References

[BS] Brodmann, Markus P., and Rodney Y. Sharp. Local cohomology: an algebraic introduction with geometric applications. Vol. **136**. Cambridge University Press, 2012.

[H] R. Hartshorne, Complete intersections in characteristic $p > 0$, Amer. J. Math. **101** (1979), no. 2, 380–383.

[L] G. Lyubeznik, A survey of problems and results on the number of defining equations, in *Commutative algebra (Berkeley, CA, 1987)*, 375–390, Math. Sci. Res. Inst. Publ., 15 Springer, New York.

[RS] H. Roloff and J. Stückrad, Bemerkungen über Zusammenhangseigenschaften und mengentheoretische Darstellung projektiver algebraischer Mannigfaltigkeiten, Wiss. Beitr. Martin-Luther-Univ. Halle-Wittenberg M **12** (1979), 125–131.

[T1] A. Thoma, Monomial space curves in $\mathbf{P}_k^3$ as binomial set theoretic complete intersections, Proc. Amer. Math. Soc. **107** (1989), no. 1, 55–61.

[T2] A. Thoma, On set-theoretic complete intersections in $\mathbf{P}_k^3$, Manuscripta Math. **70** (1991), no. 3, 261–266.

[T3] A. Thoma, On the arithmetically Cohen-Macaulay property for monomial curves, Comm. Algebra **22** (1994), no. 3, 805–821.

[T4] A. Thoma, On the equations defining monomial curves, Comm. Algebra **22** (1994), no. 7, 2639–2649.